

# Learning the Negative Binomial Distribution: Definition, Formula, and Examples

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November 8, 2025

## RECOMMENDED CITATION

Mohammed looti (2025). *Learning the Negative Binomial Distribution: Definition, Formula, and Examples*. PSYCHOLOGICAL STATISTICS. Retrieved from <https://statistics.arabpsychology.com/?p=13104>

The [negative binomial distribution](#) (NBD) is a foundational concept in [probability](#) theory and statistics, offering a robust mathematical framework for modeling sequential random events. Unlike distributions that rely on a fixed total number of trials, the NBD precisely quantifies the likelihood that a specific number of "failures" will occur before achieving a designated number of "successes" within a series of independent trials. This distribution is indispensable in scenarios where the sequence of experiments naturally terminates only upon reaching a predefined success count, rather than after a fixed total run time.

The conceptual backbone of the negative binomial distribution is the [Bernoulli trial](#). By definition, a Bernoulli trial is any random experiment characterized by two mutually exclusive outcomes, conventionally labeled "success" or "failure." Crucially, the [probability](#) of success, denoted by  $p$ , must remain absolutely constant across every repetition of the experiment, and the trials must be independent. This simple structure is the fundamental building block for most discrete statistical models.

A classic illustration involves flipping a fair coin. This action constitutes a perfect [Bernoulli trial](#). If we define "heads" as a success and "tails" as a failure, there are only two possible, discrete outcomes. For a fair coin, the probability of success is  $p = 0.5$ , and this likelihood does not change regardless of previous flips. The negative binomial distribution leverages this structure by chaining multiple independent trials together until a predetermined count of successes is ultimately achieved.

## Defining the Negative Binomial Distribution and Its PMF

The negative binomial distribution is defined by its [probability mass function](#) (PMF), which determines the probability for a random variable  $X$ . In the context of NBD,  $X$  is specifically defined as the number of failures ( $k$ ) that must occur before the  $r$ -th success is finally observed. The key differentiating factor from the binomial distribution is that the NBD fixes the stopping criterion based on the number of successes ( $r$ ), whereas the binomial distribution fixes the total number of trials ( $n$ ). This unique focus makes the NBD exceptionally suited for modeling effort, cost, or time required to meet a specific performance quota.

When a random variable  $X$  adheres to the [negative binomial distribution](#), the exact [probability](#) of experiencing precisely  $k$  failures before achieving a total of  $r$  successes can be calculated using the following formula:

$$P(X=k) = \binom{k+r-1}{k} p^r (1-p)^k$$

To correctly apply this formula, it is essential to define the required parameters. The interpretation of the calculation hinges entirely on the proper identification of these variables:

**k:** Represents the exact number of failures observed before the sequence is stopped by the  $r$ -th success.

**r:** The fixed number of successes required to terminate the sequence of trials; this is the stopping criterion.

**p:** The constant, unchanging [probability](#) of success on any single trial.

**$\binom{k+r-1}{k}$ :** This term calculates the number of [combinations](#) for selecting the  $k$  failure positions among the first  $k+r-1$  trials. Since the sequence must conclude with the  $r$ -th success, that final trial is fixed, and we only arrange the preceding outcomes.

## Step-by-Step Calculation Example

To solidify the practical application of the negative binomial formula, consider a typical sequential trial scenario. Imagine we are repeatedly flipping a fair coin, designating landing on heads as a "successful" event. Our goal is to determine the likelihood of observing exactly 6 failures (tails) before the sequence halts upon recording the 4th success (head).

This problem perfectly aligns with the NBD framework because the number of successes ( $r=4$ ) is fixed as the termination point, and we are calculating the probability associated with a specific count of preceding failures ( $k=6$ ). Since the coin is fair, the constant probability of success  $p$  is 0.5.

We structure the calculation using the identified parameters:

**k (failures):** 6

**r (successes):** 4

**p (probability of success):** 0.5

Inserting these figures into the PMF, the computation proceeds as follows:

$$P(X=6 \text{ failures}) = \binom{6+4-1}{6} * (0.5)^4 * (1-0.5)^6$$

First, we calculate the [combinations](#) term:  $\binom{9}{6} = 84$ . This value represents the 84 distinct ways that 6 failures and 3 successes can be arranged among the first 9 trials, ensuring the 10th trial is the required 4th success. Next, we determine the probability components:  $(0.5)^4 = 0.0625$  and  $(0.5)^6 = 0.015625$ . Multiplying these elements yields:  $(84) * (0.0625) * (0.015625) = \mathbf{0.08203}$ . Therefore, there is an 8.2% chance of needing exactly 6 tails before finally achieving the 4th head.

## Essential Properties: Mean and Variance

Interpreting the central tendency and dispersion of the [negative binomial distribution](#) is crucial for practical predictive modeling and statistical inference. The two primary descriptive statistics are the expected value (or [mean](#)) and the [variance](#). These properties enable researchers to estimate the typical outcome and the anticipated degree of variability in the number of required failures.

The [mean](#), denoted  $E$ , provides the expected number of failures ( $k$ ) we anticipate before reaching the target  $r$  successes. This value is directly proportional to the required success count  $r$  and inversely proportional to the probability of success  $p$ . The standard formula for the expected number of failures is: **Mean (Expected Failures) =  $r(1-p) / p$** . A lower probability of success  $p$  logically necessitates a higher expected number of failures to achieve the fixed target  $r$ .

The [variance](#),  $Var$ , quantifies the spread or volatility around the expected value. A large variance suggests that the actual number of failures observed in different repetitions of the experiment is likely to deviate significantly from the [mean](#). The variance for the number of failures before achieving  $r$  successes is calculated as: **Variance (Var) =  $r(1-p) / p^2$** . Notice the squared term in the denominator; this indicates that the variance grows rapidly as the probability of success  $p$  decreases, making outcomes in low-success-rate environments inherently more unpredictable.

Applying these properties to our fair coin flip example (where the target  $r=4$  and  $p=0.5$ ): The [mean](#) number of failures (tails) expected before achieving 4 successes is calculated as: **Mean =  $(4 * (1 - 0.5)) / 0.5 = 2 / 0.5 = 4$** . Subsequently, the [variance](#) is calculated as: **Variance =  $(4 * (1 - 0.5)) / (0.5)^2 = 2 / 0.25 = 8$** .

## Diverse Applications of the Negative Binomial Distribution

The [negative binomial distribution](#) is a highly flexible statistical tool with significant relevance across numerous technical and scientific disciplines. Its utility shines when modeling processes that continue until a specific quota or performance goal is satisfied, extending its use far beyond simple pass/fail binomial analysis.

In fields such as **quality control and reliability engineering**, the NBD is used to predict the required inputs or resources needed to meet a production quota. For instance, if a manufacturer requires 10 perfectly functional integrated circuits (successes), the NBD can estimate the expected number of defective circuits (failures) that will be produced along the way. This calculation provides crucial data for informing cost analysis, managing inventory, and optimizing production schedules.

**Epidemiology and biological research** frequently utilize the NBD, especially when modeling rare events or complex sample collection efforts. If researchers need to recruit a specific number of patients who possess a rare genetic marker (successes), the NBD can estimate the total number of individuals that must be screened (failures) before the recruitment target is met. This predictive power is invaluable for clinical trials planning and budget estimation.

Furthermore, in **economics and finance**, the negative binomial model is often the preferred choice over the Poisson distribution for analyzing count data, particularly when the data exhibits overdispersion. Overdispersion occurs when the data's variability ([variance](#)) significantly exceeds its average ([mean](#)). Examples include modeling the frequency of major market crashes, the

number of large insurance claims, or the count of high-volume trading days. The NBD naturally incorporates this higher variance, offering a more stable and robust fit for volatile real-world financial data.

## Practice Problems and Calculated Results

To properly test and reinforce your understanding of the negative binomial distribution, analyze and solve the following practical problems. Each scenario requires the accurate identification of the parameters ( $k$ ,  $r$ , and  $p$ ) and the subsequent application of the probability mass function.

**Note:** For the sake of efficiency and precision in obtaining the final numerical results, these calculations reference the use of specialized statistical software or an online negative binomial distribution calculator.

### Problem 1: Coin Flip Sequence Analysis

**Question:** Suppose we are flipping a fair coin, defining landing on heads as a "successful" outcome. What is the [probability](#) of experiencing exactly 3 failures (tails) before achieving a total of 4 successes (heads)?

**Answer:** Applying the Negative Binomial Distribution Calculator with the parameters  $k = 3$  failures,  $r = 4$  successes, and  $p = 0.5$ , the calculated probability  $P(X=3)$  is found to be: **0.15625**.

### Problem 2: Sales Success Rate Modeling

**Question:** Consider a sales campaign where securing a purchase is deemed a "success." Historical data indicates the probability that any given person will buy the product is  $p = 0.4$ . What is the probability of experiencing 8 unsuccessful attempts (failures) before we achieve a total of 5 successful sales?

**Answer:** Using the Negative Binomial Distribution Calculator with  $k = 8$  failures,  $r = 5$  successes, and  $p = 0.4$ , the resulting probability  $P(X=8)$  is approximately: **0.08514**.

### Problem 3: Die Roll Target Achievement

**Question:** We are rolling a standard six-sided die, defining a "successful" roll as landing specifically on the number 5. The probability of this success on any roll is  $1/6$ , or approximately  $p = 0.167$ . What is the probability of experiencing exactly 4 failures before we achieve a total of 3 successes?

**Answer:** Utilizing the Negative Binomial Distribution Calculator with  $k = 4$  failures,  $r = 3$  successes, and  $p = 0.167$ , the calculated probability  $P(X=4)$  is found to be: **0.03364**.